Math 10B with Professor Stankova
Worksheet, Discussion #14; Tuesday, 3/12/2019
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1 Expected Value

1.1 Concepts

1.

Distribution	PMF	$E(X)$
Uniform	If $\#R(X) = n$, then $f(x) = \frac{1}{n}$ for all $x \in R(X)$.	$\sum_{i=1}^{n} \frac{x_i}{n}$
Bernoulli Trial	$f(0) = 1 - p$, $f(1) = p$	p
Binomial	$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np
Geometric	$f(k) = (1-p)^k p$	$\frac{1-p}{p}$
Hyper-Geometric	$f(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$	$\frac{nm}{N}$
Poisson	$f(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	λ

The **Expected Value** is the weighted average of all the values the random variables can take on. By definition, it is:

$$E[X] = \sum x_i f(x_i).$$

It satisfies some properties:

- $E[c] = c$
- $E[cX] = cE[X]$
- $E[X + Y] = E[X] + E[Y]$ for **all** random variables
- $E[XY] = E[X]E[Y]$ for **independent** random variables.

1.2 Examples

2. I flip a fair coin 5 times. What is the expected number of heads I flip?

Solution: This is a binomial distribution with $p = \frac{1}{2}$ and $n = 5$. Then $E(X) = 5/2$.

3. I roll two fair 6 sided die. What is the expected value of their product?

Solution: Let X be the first value I roll and Y be the second. The rolls are independent and so $E[XY] = E[X]E[Y] = 3.5 \cdot 3.5 = 12.25$.

4. I randomly rearrange 10 people of different heights in a line. Let X be the number of people in the right sorted order (person i is the ith tallest person). What is $E[X]$?

Solution: Let X_1 be 1 if the first person is the shortest and 0 otherwise, X_2 be 1 if the second person is the second shortest and 0 otherwise, all the way up to X_{10}. Then $X = X_1 + X_2 + \cdots + X_{10}$ and we have that

$$E[X] = E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{10}].$$

Each X_i can be thought of as a Bernoulli trial with probability of success $\frac{1}{10}$ so $E[X_i] = \frac{1}{10}$ and

$$E[X] = \frac{1}{10} + \cdots + \frac{1}{10} = 1.$$

1.3 Problems

5. True **FALSE** The expected value of a random variable X is the value such that the PMF at that point is the largest.

Solution: The PMF at that point may be 0! For example, there expected number of heads when flipping 5 coins is 2.5 but we cannot flip 2.5 heads.

6. True **FALSE** The expected value of a random variable X always exists.

Solution: This is false as seen in the previous homework where the expected value was infinite.

7. True **FALSE** We have that $E[X^2] = E[X \cdot X] = E[X]E[X]$.

Solution: X and X are not independent from each other.

8. **TRUE** False If $a \leq X \leq b$ (a is the smallest X can be and b is the largest), then $a \leq E[X] \leq b$.

Solution: You can think of this as if a die roll is in between 1 and 6, then the expected value can't be negative or bigger than 6.

9. While pulling out of a box of cookies, what is the expected number of cookies I have to pull out before I pull out an oatmeal raisin if 20% of cookies are oatmeal raisin?

Solution: This is a geometric distribution because I am counting the number of cookies I have to pull out before a success. The probability of success is $20\% = p = 1/5$. So the expected number of cookies I have to pull out is $\frac{1-p}{p} = 4$.

10. What is the expected number of aces I have when I draw 5 cards out of a deck?

Solution: Drawing cards out of a deck without replacement is the hypergeometric distribution. There are $N = 52$ cards total and $m = 4$ aces total. Then, we pull out $n = 5$ cards and so the expected number of aces is $\frac{mn}{N} = \frac{20}{52}$.

11. I am rolling two die and I stop when I roll snake eyes (2 1's). What is the expected number of times I have to roll the die?

Solution: This is a geometric distribution because I roll until a success. The probability of success is $p = \frac{1}{36}$. Looking at the table, I am expected to have $\frac{1-p}{p} = 35$ failures and so I roll the die $35 + 1 = 36$ times total.

12. The number of lightning strikes during a thunderstorm is given by a Poisson distribution with expected value 10. What is the probability that there are 5 strikes in the latest storm?

Solution: Since the expected value is λ, we have that $\lambda = 10$. Now the probability of 5 strikes is

$$f(5) = \frac{10^5 e^{-10}}{5!}.$$

13. In a safari, safari-keepers have caught and tagged 300 rhinos. On a safari, out of the 15 different rhinos you see, there are 5 of them expected to be tagged. How many rhinos are there at the safari?

Solution: This is a hyper-geometric distribution because out of the N rhinos total and $m = 300$ tagged rhinos, you see that $n = 15$ rhinos that you see, there are 5 of them expected to be tagged. So $5 = E(X) = \frac{mn}{N} = \frac{300 \cdot 15}{N}$. So $N = \frac{300 \cdot 15}{5} = 900$.

14. (Challenge) In a class of 30 students, I split them up into 6 groups of 5 on Tuesday. Today, Thursday, I split them up again randomly. What is the expected number of people in your new group were in your old group on Tuesday?

Solution: We can think of this as a hypergeometric distribution where day 1, we "tag" or mark the 4 students that were in your group. Then on day 2, out of the 29 other students, you want to select 4 of them without replacement to be in your group. This is a hyper-geometric distribution with $N = 29, n = 4$. Then $m = 4$ because there are 4 students that were in your group before. So, the expected number of people in your new group who were in your old group is $\frac{mn}{N} = \frac{4 \cdot 4}{29} = \frac{16}{29}$.